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1999 J. Phys. A: Math. Gen. 32 1241

(http://iopscience.iop.org/0305-4470/32/7/013)

J. Phys. A: Math. Gen. **32** (1999) 1241–1249. Printed in the UK PII: S0305-4470(99)95891-2

Dynamical symmetries, non-Cartan symmetries and superintegrability of the n-dimensional harmonic oscillator

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Received 14 July 1998, in final form 27 November 1998

Abstract. The theory of dynamical but non-Cartan (or non-Noether) symmetries is studied using the symplectic formalism approach. It is shown that the superintegrability of the n-dimensional non-isotropic harmonic oscillator is directly related to the existence of dynamical but non-Cartan symmetries.

1. Introduction

A superintegrable system is a system that is integrable (in the Liouville–Arnold sense) and that, in addition to this, possesses more constants of motion than degrees of freedom [1–21]. If the number N of independent constants takes the value $N = 2n - 1$ (where n is the number of degrees of freedom) then the system is called maximally superintegrable. There are three classic and well known cases of this very particular class of systems, namely, the free particle, the Kepler problem, and the harmonic oscillator with rational frequencies. In all these three cases it is known that all the orbits become closed for the case of bounded motions. This high degree of regularity (the existence of periodic motions) is a consequence of their superintegrable character. An important point to note is that these three systems are superintegrable not only in the standard case of $n = 3$ but also in the general case of an arbitrary number n of degrees of freedom. More recently the existence of other less simple superintegrable n-dimensional systems such as the Calogero–Moser system [4, 21], the Smorodinsky–Winternitz system [11], or the hyperbolic Calogero–Sutherland–Moser model [20] has been proved.

According to the Noether approach to the dynamics, the existence of integrals of motion is related to the theory of symmetries. Consequently superintegrable systems must be considered as systems endowed with a rich variety of symmetries. The purpose of this paper is to present a study of the superintegrability of the n-dimensional harmonic oscillator using the geometric formalism and the theory of symmetries as an approach.

The paper is organized as follows. In section 2 we present a detailed (but non-geometric) discussion of the particular $n = 2$ case. Section 3 is devoted to the geometric study of the theory of symmetries (dynamical, Cartan and non-Cartan symmetries) and then section 4 states the direct relation between the superintegrability of the harmonic oscillator and the existence of dynamical but non-Cartan symmetries.

0305-4470/99/071241+09$19.50

2. Superintegrability of the harmonic oscillator

The $n = 2$ harmonic oscillator

$$H_{\mathrm{HO}} = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}({\omega_1}^2 q_1^2 + {\omega_2}^2 q_2^2)$$

is a trivially integrable system, since it is a direct sum of sytems with one degree of freedom and, therefore, it has the two one-degree of freedom energies, E_1 and E_2, as involutive integrals. If the oscillator is isotropic then it has the angular momentum as an additional integral of motion [8, 20, 22]. If the oscillator is non-isotropic then the angular momentum is not preserved, but in the very particular case in which the quotient of the two frequencies is rational the system has a third additional nonlinear integral. In geometric terms the phase space is foliated by tori and every integral curve is a curve with constant slope on a torus. The slope of the curve is determined by the ratio ω_2/ω_1. Thus, if this ratio is irrational the corresponding curve will be dense on the torus. If this ratio is rational then the orbit becomes closed and the motion will be periodic.

Let us denote the following two functions by $T_i = T_i(q, p)$, $i = 1, 2$:

$$T_i = \sin^{-1} z_i \qquad z_i = \frac{\omega_i q_i}{\sqrt{\omega_i^2 q_i^2 + p_i^2}}$$

(for ease of notation we write all the indices as subscripts). Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} T_i = \omega_i \qquad i = 1, 2.$$

Let us denote by I_i^t, $i = 1, 2$, the two functions

$$I_i^t = \sin^{-1} z_i - \omega_i t \qquad i = 1, 2.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} I_i^t = 0 \qquad i = 1, 2.$$

So the functions I_i^t, $i = 1, 2$, are time-dependent constants of motion. Moreover the function I_{12} defined by

$$I_{12} = \omega_2 \sin^{-1} z_1 - \omega_1 \sin^{-1} z_2$$

is also a constant of motion.

Notice that the functions T_i are defined out of the origin and up to 2π. Notice also that every I_i^t is a single-dependent degree of freedom function. Concerning I_{12}, it must be considered as introducing a coupling between the two degrees of freedom. In fact it can be written as

$$I_{12} = I_{12}^0 + I_{12}^a \qquad I_{12}^a = 2\pi\,(\omega_2 k_1 - \omega_1 k_2) \qquad k_1, k_2 = 0, \pm 1, \pm 2, \ldots$$

where I_{12}^0 is the fundamental value and I_{12}^a represents the ambiguity. If ω_2/ω_1 is rational this ambiguity (that is a multiple of 2π) can be removed from I_{12}^0; only in this case I_{12} is a well-defined function.

This integral I_{12} will lead to the angular momentum for the isotropic case, and to the corresponding nonlinear constant for the non-isotropic rational case. If we make use of the complex logarithmic function

$$\sin^{-1} z_i = (-\mathrm{i})\, \log\left[\frac{\mathrm{i}\omega_i q_i + p_i}{\sqrt{\omega_i^2 q_i^2 + p_i^2}}\right] \qquad i = 1, 2$$

then the integral I_{12} can be transformed into the following expression:

$$J_{12} = (p_1 + \mathrm{i}\omega_1 q_1)^{\omega_2}(p_2 - \mathrm{i}\omega_2 q_2)^{\omega_1}$$
$$= (\omega_1^2 q_1^2 + p_1^2)^{\omega_2/2}(\omega_2^2 q_2^2 + p_2^2)^{\omega_1/2} \exp\big[\mathrm{i}(\omega_2\theta_1 - \omega_1\theta_2) + 2\pi\,\mathrm{i}(\omega_2 k_1 - \omega_1 k_2)\big].$$

Only if ω_2/ω_1 is rational is J_{12} a well-defined function. If $\omega_2/\omega_1 = m/n$ then $\mathrm{Im}\,(J_{12})$ and $\mathrm{Re}\,(J_{12})$ are polynomials in the momenta of degree $m+n-1$ and $m+n$, respectively. Next, we give the expressions for these two constants for the first three cases:

(i) The isotropic case $\omega_2 = \omega_1 = \omega$

$$\mathrm{Re}\,(J_{12}) = p_1 p_2 + \omega^2 q_1 q_2$$
$$\mathrm{Im}\,(J_{12}) = q_2 p_1 - q_1 p_2.$$

$\mathrm{Im}\,(J_{12})$ is just the angular momentum, and $\mathrm{Re}\,(J_{12})$ is the non-diagonal component of the Fradkin tensor [23].

(ii) The non-isotropic case with $\omega_2 = 2\omega$, $\omega_1 = \omega$

$$\mathrm{Re}\,(J_{12}) = p_1^2 p_2 - \omega^2(q_1 p_2 - 4q_2 p_1)q_1$$
$$\mathrm{Im}\,(J_{12}) = (q_1 p_2 - q_2 p_1)p_1 + \omega^2 q_1^2 q_2.$$

(iii) The non-isotropic case with $\omega_2 = 3\omega$, $\omega_1 = \omega$

$$\mathrm{Re}\,(J_{12}) = p_1^3 p_2 - 3\omega^2(q_1 p_2 - 3q_2 p_1)q_1 p_1 - 3\omega^4 q_1^3 q_2$$
$$\mathrm{Im}\,(J_{12}) = 3(q_1 p_2 - q_2 p_1)p_1^2 - \omega^2(q_1 p_2 - 9q_2 p_1)q_1^2.$$

Consequently, in this particular $n = 2$ case, superintegrability arises as a consequence of the existence of two functions T_i, $i = 1, 2$, that can be considered as generating the additional integrals of motion. First they give rise to the two functions I_i^t, $i = 1, 2$, in such a way that we obtain the time-dependent set $\{E_1, E_2; I_1^t, I_2^t\}$. Second we obtain, if a certain property is satisfied, the time-independent function I_{12}. In this case the system is superintegrable with the time-independent set $\{E_1, E_2; I_{12}\}$ as a set of $N = 3$ fundamental constants.

We close this section with the following observations.

(a) A time-independent system can be endowed with time-dependent constants of motion. The classical example is the $n = 1$ free particle that possesses $I^t = q - pt$ as an integral. Nevertheless this situation is rather unusual and all the known cases have a very simple dependence on time (e.g., they are linear functions of t). Concerning the two functions I_1^t, I_2^t, they can be rewritten as the argument of a constant complex function

$$I_i^t = \arg(J_i^t) \qquad J_i^t = (p_i + \mathrm{i}\omega_i q_i)\mathrm{e}^{-\mathrm{i}\omega_i t} \quad \frac{\mathrm{d}}{\mathrm{d}t} J_i^t = 0 \quad i = 1, 2.$$

(b) Integrability in the Liouville–Arnold sense (superintegrability) of a time-independent Hamiltonian system must be related to a set of n $(2n-1)$ time-independent integrals. This means that the existence of I_{12} must be considered as more fundamental than the existence of the pair $\{I_1^t, I_2^t\}$.

(c) The two functions I_1^t, I_2^t, are well defined as integrals of motion and the problem of the quotient of the frequencies does not affect them since every I_i^t depends only on its own degree of freedom. That is, I_i^t (or J_i^t) depends on ω_i and ignores the value of ω_j, $i \neq j$. Consequently the time-dependent set of $N = 4$ integrals given by $\{E_i; I_i^t\}$ is well defined regardless of the frequencies.

(d) I_{12} couples the two degrees of freedom and depends of the relation between ω_2 and ω_1. As stated above, I_{12} is a well-defined function only if the quotient ω_2/ω_1 is rational.

3. Dynamical symmetries and non-Cartan symmetries

The Hamiltonian phase space is the $2n$-dimensional symplectic manifold (T^*Q, ω_0) where T^*Q is the cotangent bundle of the configuration space Q [24–27] and ω_0 is the canonical symplectic structure

$$\omega_0 = -\mathrm{d}\theta_0 \qquad \theta_0 = p_j\, \mathrm{d}q_j.$$

The dynamics is represented by the Hamiltonian vector field $\Gamma_H \in \mathfrak{X}(T^*Q)$ of the Hamiltonian function H with respect to ω_0

$$i(\Gamma_H)\omega_0 = \mathrm{d}H \qquad \omega_0 = \mathrm{d}q_j \wedge \mathrm{d}p_j$$

(summation over the index j is understood).

There are two different ways of approaching the theory of symmetries: (1) the symmetries of the dynamical vector field, and (2) the symmetries of the Hamiltonian system (T^*Q, ω_0, H).

In differential geometric terms, a dynamical symmetry of the dynamics is a vector field X on T^*Q such that $[X, \Gamma_H] = 0$ [28]. If X is the complete lift to T^*Q of a vector field previously defined on Q then X is a Lie symmetry (Lie symmetries are projectable onto Q).

A Noether symmetry is a vector field X defined on the configuration space Q such that its complete lift X^t to T^*Q satisfies the following two properties: (i) The Lie derivative of θ_0 with respect to X^t is exact (this means that X^t is a symmetry of the symplectic form), (ii) X^t is a symmetry of the Hamiltonian, that is, $X^t(H) = 0$. A Cartan symmetry is a vector field Y that is directly defined on T^*Q and that also satisfies the above two properties, i.e. (i) $\mathcal{L}_Y\theta_0$ is exact and consequently $\mathcal{L}_Y\omega_0 = 0$, and (ii) Y is a symmetry of H. It is clear that the idea of Cartan symmetry is just an extension of the idea of Noether symmetry and, conversely, a Noether symmetry can be considered as a Cartan symmetry that is projectable. Notice that if a symmetry is of Noether class then it generates a one-parameter group of point transformations. The transformations generated by Cartan vector fields are more general (momentum-dependent transformations in the usual language). We also notice that some other authors use the name Noether for both types of symmetries, and the name Cartan for the symmetries of the time-dependent one-form $\Theta_H = \theta_0 - H\,\mathrm{d}t$

Cartan symmetries are important because of the two following properties: (1) every Cartan symmetry determines an integral of motion I for H, and, (2) every Cartan symmetry is a symmetry of the dynamics.

The important point is that the converse of (2) is not true; so Cartan symmetries are in fact a subclass of the dynamical symmetries. The theory of symmetries have been extensively analysed but, because of property (1), most of the studies have focused on the Noether theorem and on the Cartan symmetries. So the existence of Hamiltonians with 'dynamical but non-Cartan symmetries' can probably be considered as a rather unusual situation. Nevertheless, as we will prove, this peculiar situation is the key to understanding superintegrability of the harmonic oscillator.

The following proposition is of great importance for our geometric approach.

Proposition 1. *Let H be a Hamiltonian function and Γ_H the associated Hamiltonian vector field. Suppose that X is a dynamical but non-Cartan symmetry. Then*
(i) The function $X(H)$ is a constant of motion.
(ii) The dynamical vector field Γ_H is a bi-Hamiltonian system.

Proof. (i) If X is a dynamical symmetry then $[X, \Gamma_H] = 0$ and we have

$$i([X, \Gamma_H])\,\omega_0 = \mathcal{L}_X[i(\Gamma_H)\,\omega_0] - i(\Gamma_H)[\mathcal{L}_X\omega_0] = 0.$$

We have

$$\mathcal{L}_X[i(\Gamma_H)\,\omega_0] = \mathcal{L}_X(\mathrm{d}H) = \mathrm{d}[X(H)]$$

and therefore

$$\mathrm{d}[X(H)] = i(\Gamma_H)[\mathcal{L}_X\omega_0].$$

Hence we arrive at

$$\Gamma_H[X(H)] = i(\Gamma_H)i(\Gamma_H)[\mathcal{L}_X\omega_0] = 0.$$

Consequently the function $X(H)$ is a constant of motion.

(ii) Let us denote by ω_X the two-form $\omega_X = \mathcal{L}_X\omega_0$. Notice that ω_X, that in the general case $\omega_X \neq \omega_0$, can be even a non-symplectic two-form (that is, with a non-trivial kernel). The vector field Γ_H is the solution of the following two equations:

$$i(\Gamma_H)\,\omega_0 = \mathrm{d}H \qquad \text{and} \qquad i(\Gamma_H)\,\omega_X = \mathrm{d}[X(H)].$$

So Γ_H is a bi-Hamiltonian vector field. □

We illustrate this situation by the following example. The Hamiltonian vector field Γ_{HO} of the $n = 2$ isotropic oscillator is given by

$$\Gamma_{\mathrm{HO}} = p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2} - \omega^2 q_1\frac{\partial}{\partial p_1} - \omega^2 q_2\frac{\partial}{\partial p_2}.$$

Then the non-Hamiltonian vector field

$$X = q_2\frac{\partial}{\partial q_1} + q_1\frac{\partial}{\partial q_2} + p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2}$$

is a non-Cartan dynamical symmetry

$$[X, \Gamma_{\mathrm{HO}}] = 0 \qquad X(H_{\mathrm{HO}}) = H_X \qquad H_X = 2\,[p_1p_2 + \omega^2 q_1q_2].$$

The new symplectic structure is

$$\omega_X = \mathcal{L}_X\omega_0 = 2\,[dq_1 \wedge dp_2 + dq_2 \wedge dp_1]$$

and we obtain the following bi-Hamiltonian formulation

$$i(\Gamma_{\mathrm{HO}})\,\omega_0 = \mathrm{d}H_{\mathrm{HO}} \qquad \text{and} \qquad i(\Gamma_{\mathrm{HO}})\,\omega_X = \mathrm{d}H_X.$$

The next proposition considers the case of a vector field that, although is a symmetry of the symplectic structure, does not preserve the Hamiltonian.

Proposition 2. *Let X_T be a Hamiltonian vector field with the function T as Hamiltonian. If X_T is a dynamical but non-Cartan symmetry then the function $X_T(H) = \alpha \neq 0$ is a numerical constant.*

Proof. If X_T is a dynamical symmetry then $[X_T, \Gamma_H] = 0$ so we have

$$\mathcal{L}_{X_T}[i(\Gamma_H)\,\omega_0] - i(\Gamma_H)[\mathcal{L}_{X_T}\omega_0] = 0.$$

Since X_T is Hamiltonian vector field we have $\mathcal{L}_{X_T}\omega_0 = 0$. Therefore we arrive at

$$\mathcal{L}_{X_T}[i(\Gamma_H)\,\omega_0] = \mathrm{d}[X_T(H)] = 0.$$

Hence the function $X_T(H) = \alpha$ must be a numerical constant. □

To prove the next proposition we will make use of the time-dependent Hamiltonian formalism. The time-dependent phase space is now the manifold $M = T^*Q \times \mathbb{R}$, and the fundamental geometric structure on M is the exact two-form of contact Ω_H defined as

$$\Omega_H = \omega_0 + \mathrm{d}H \wedge \mathrm{d}t.$$

The dynamics is given by the unique vector field $\widetilde{\Gamma}_H$ defined on $M = T^*Q \times \mathbb{R}$ as solution of the following two equations

$$i(\widetilde{\Gamma}_H)\Omega_H = 0 \qquad i(\widetilde{\Gamma}_H)\mathrm{d}t = 1.$$

In coordinates $\widetilde{\Gamma}_H$ takes the form

$$\widetilde{\Gamma}_H = \left(\frac{\partial H}{\partial p_j}\right)\frac{\partial}{\partial q_j} - \left(\frac{\partial H}{\partial q_j}\right)\frac{\partial}{\partial p_j} + \frac{\partial}{\partial t}.$$

Here $\widetilde{\Gamma}_H$ is usually called [24] the suspension of Γ_H.

At this point we make the following observation. If Γ is a differentiable vector field on an m-dimensional manifold M then the maximun number of independent functions which are invariant along the integral curves of Γ is $N = m - 1$. So Γ_H can have a maximun number of $N = 2n - 1$ time-independent integrals since it is defined on (T^*Q, ω_0). However, $\widetilde{\Gamma}_H$, as defined in $(T^*Q \times \mathbb{R}, \Omega_H)$, can admit $N = 2n$ independent time-dependent integrals.

Proposition 3. *Let T be a time-independent function, X_T its Hamiltonian vector field, and suppose that X_T is a dynamical but non-Cartan symmetry. Then the time-dependent function I^t defined as*

$$I^t = T + X_T(H)\,t$$

is a time-dependent constant of motion.

Proof. The inner product of X_T with Ω_H is given by

$$\begin{aligned} i(X_T)\,\Omega_H &= i(X_T)\,\omega_0 + [i(X_T)\,\mathrm{d}H]\,\mathrm{d}t \\ &= \mathrm{d}T + X_T(H)\,\mathrm{d}t = \mathrm{d}(T + \alpha t) \end{aligned}$$

where we have taken into account that α is a numerical constant. Therefore

$$\begin{aligned} \widetilde{\Gamma}_H(T + \alpha t) &= i(\widetilde{\Gamma}_H) i(X_T)\,\Omega_H \\ &= -\,i(X_T) i(\widetilde{\Gamma}_H)\,\Omega_H = 0 \end{aligned}$$

and the proposition is proved. □

Now suppose that not one but two different dynamical symmetries X_i, $i = 1, 2$, of such a class for the same dynamical vector field Γ_H, and suppose that both X_i are Hamiltonian vector fields. Let us denote by T_i, $i = 1, 2$, the two independent Hamiltonian functions. Then we have the following property:

Proposition 4. *The vector field X_{12} defined as*

$$X_{12} = \alpha_2 X_1 - \alpha_1 X_2$$

is a Cartan symmetry.

Proof. The vector field X_{12} is a symmetry of the Hamiltonian

$$\begin{aligned} X_{12}(H) &= \alpha_2 X_1(H) - \alpha_1 X_2(H) \\ &= \alpha_2\alpha_1 - \alpha_1\alpha_2 = 0. \end{aligned}$$

X_{12} is also a symmetry of the symplectic form since it is linear combination of Hamiltonian vector fields. Thus X_{12} is a Cartan symmetry. □

Moreover, the inner product of X_{12} with the symplectic form is given by

$$\begin{aligned} i(X_{12})\omega_0 &= \alpha_2 i(X_1)\,\omega_0 - \alpha_1 i(X_2)\,\omega_0 \\ &= \alpha_2\,\mathrm{d}T_1 - \alpha_1\,\mathrm{d}T_2 = \mathrm{d}(\alpha_2 T_1 - \alpha_1 T_2). \end{aligned}$$

Thus the vector field X_{12} determines the function $I_{12} = \alpha_2 T_1 - \alpha_1 T_2$ as its associated (time-independent) constant of motion.

Thus, according to this proposition, every pair of two Hamiltonian vector fields which are dynamical but non-Cartan symmetries determine one Cartan symmetry.

Notice that systems possessing these rather special Cartan symmetries (arising from the pairing of two dynamical symmetries) must necessarily be systems with time-dependent constants. This means a restriction since, as stated above, the number of known time-independent systems possessing time-dependent integrals seems to be very reduced. In any case, the existence of such a coupling between the two function (T_1, T_2) gives rise to a new integral of motion.

Crampin proved [28] that, in the tangent bundle TQ, the existence of a 'dynamical but non-Cartan symmetry' which is projectable (i.e. a non-Noether Lie symmetry) is related to the existence of an alternative Lagrangian. Alternative Lagrangians are related to new symplectic structures and to non-Noether constants of motion. So there is a relation between non-Cartan symmetries and alternative ways of obtaining integrals of motion. We point out that the result given by proposition 4 do not require the condition of projectability.

For the $n = 2$ case, if H is integrable with involutive integrals I_1, I_2, the existence of two generating functions (T_1, T_2) leads to a new function I_{12}. If $\mathrm{d}I_1 \wedge \mathrm{d}I_2 \wedge \mathrm{d}I_{12} \neq 0$ then I_{12} is independent and H is superintegrable. The important point is that this superintegrability must be considered as a consequence of the previous existence of the pair (T_1, T_2).

The generalization of this last result to the general n-dimensional case is as follows. Let the n-dimensional Hamiltonian system (T^*Q, ω_0, H) be integrable with n independent and involutive integrals I_k, $k = 1, 2, \ldots, n$, arising from n Cartan (or Noether) symmetries. If this system has n independent 'dynamical but non-Cartan symmetries' represented by the Hamiltonian vector fields X_r, $r = 1, 2, \ldots, n$, then the system has, in addition to the n fundamental integrals, other

(i) n functionally independent time-dependent integrals I_r^t;
(ii) $\frac{1}{2}n(n-1)$ additional time-independent integrals I_{rs}.

If the vector fields X_r are independent then so are the n Hamiltonian functions T_r, $r = 1, 2, \ldots, n$, and, because of this, the system of $n - 1$ functions I_{rr+1} is also independent. Moreover the independence of the set $\{I_r, T_r\}$ is a sufficient condition for the independence of $\{I_r, I_{rr+1}\}$. In this case the system is not only integrable but also superintegrable.

4. Superintegrability of the harmonic oscillator: II. A geometric approach

The n-dimensional harmonic oscillator

$$H_{\mathrm{HO}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \frac{1}{2}\sum_{j=1}^{n} \omega_j^2 q_j^2$$

is trivially integrable with the n one-degree of freedom energies as the n fundamental involutive integrals, i.e. $I_j = E_j$, $j = 1, 2, \ldots, n$.

Let us denote by $T_r = T_r(q, p)$, $r = 1, 2, \ldots, n$, the following n functions:

$$T_r = \sin^{-1} z_r \qquad z_r = \frac{\omega_r q_r}{\sqrt{\omega_r^2 q_r^2 + p_r^2}}.$$

Notice that every T_r is defined up to a multiple of 2π. So they must be considered as functions with values in S^1. In any case the associated vector fields X_r, $r = 1, 2, \ldots, n$, are given by

$$X_r = -\Omega_r\, X_r^0 \qquad \Omega_r = \frac{\omega_r}{\omega_r^2 q_r^2 + p_r^2} \qquad X_r^0 = q_r\,\frac{\partial}{\partial q_r} + p_r\,\frac{\partial}{\partial p_r}$$

(no summation over the index r). We have the following properties: (i) X_r is a symmetry of the symplectic form ω_0 since X_r is Hamiltonian, (ii) X_r is not a symmetry of the Hamiltonan since $X_r(H_{\rm HO}) = -\omega_r$. So X_r is neither a Noether nor a Cartan symmetry. The important point is that X_r it is a symmetry of the dynamics. This last property, that can be proved by direct calculus, can also be seen in the two following ways.

(1) The Lie bracket of two Hamiltonian vector fields is (up to a minus sign) the Hamiltonian vector field of the Poisson bracket of the two Hamiltonians. So we have

$$[X_r, \Gamma_H] = -\, X_{\{T_r, H\}}$$

and

$$\{T_r, H_{\rm HO}\} = i(X_r) i(\Gamma_H)\, \omega_0 = X_r(H_{\rm HO}) = -\omega_r.$$

Since the Hamiltonian vector field of a numerical constant vanishes, it follows that $[X_r, \Gamma_H] = 0$.

(2) The second alternative approach is to prove that the vector field $[X_r, \Gamma_H]$ is in the kernel of the symplectic form

$$i([X_r, \Gamma_H])\, \omega_0 = \mathcal{L}_{X_r}[i(\Gamma_H)\, \omega_0] - i(\Gamma_H)[\mathcal{L}_{X_r} \omega_0] = {\rm d}X_r(H_{\rm HO}) = -{\rm d}\omega_r = 0.$$

Since ω_0 is regular, the kernel is trivial and we arrive at $[X_r, \Gamma_H] = 0$.

Thus every vector field $X_r = -\Omega_r X_r^0$ is a dynamical but non-Cartan symmetry of $H_{\rm HO}$. Consequently we can apply the geometric formalism studied in section 3.

(i) Every non-Cartan symmetry X_r, $r = 1, 2, \ldots, n$, determines, in a direct way, the time-dependent function $I_r^t = T_r - \omega_r t$ as associated constant of motion. Hence $H_{\rm HO}$ has a time-dependent family of $N = 2n$ independent integrals given by $\{ E_r; I_r^t \}$.

(ii) The n functions T_r belong to the set $\mathcal{F}(T^*Q, S^1)$ of functions of T^*Q to S^1. This set is is a **Z**-modulus (not a vector space) and, because of this, linear combinatios of functions in this set are well defined only if the coefficients are integers. Thus if $\omega_r/\omega_s = m_r/n_s$ then everyone of the vector fields $X_{rs} = \omega(m_s X_r - n_r X_s)$, $r, s = 1, 2, \ldots, n$, is a Cartan symmetry with the function $I_{rs} = \omega(m_s T_r - n_r T_s)$ as Hamiltonian function. Hence $H_{\rm HO}$ has, in the particular rational case, the time-independent family of functions $\{ E_r; I_{rs} \}$ as a set of integrals of motion.

The **Z**-modulus property can also be approached as follows. The functions T_r are multivaluated but their differentials $\beta_r = {\rm d}T_r$ are well-defined one-forms and determine locally Hamiltonian vector fields. Although the exclusion of the origin makes the manifold non-contractible (topologically non-trivial), the closed one forms β_r are of integer class (mod 2π). Thus only integer combinations generate again closed one forms of integer class.

Notice that the functions I_{rs} are antisymmetric in the two indices, i.e. $I_{sr} = -I_{rs}$, so the total number of elements in the family I_{rs} is $(1/2)n(n-1)$. This means an excessive number of integrals, but one can choose, inside this family, a more reduced one-parameter subfamily with just $n-1$ independent functions. An appropriate subfamily is I_{rr+1} that couples every degree of freedom with the following one. The independence of the family $\{T_r; r = 1, \ldots, n\}$ implies the independence of $\{I_{rr+1}; r = 1, \ldots, n-1\}$, and the property ${\rm d}E_r \wedge {\rm d}E_s \wedge {\rm d}I_{rs} \neq 0,\ r \neq s$, implies the independence of the total set $\{ E_r; I_{rr+1} \}$. Moreover, in this case the $(2n)$-form Ω defined as $\Omega = {\rm d}E_1 \wedge \cdots \wedge {\rm d}E_n \wedge {\rm d}T_1 \wedge \cdots \wedge {\rm d}T_n$ is a volume form.

Thus the superintegrability of the n-dimensional harmonic oscillator with rational frequencies is a consequence of the existence of n dynamical but non-Cartan symmetries.

Finally we notice that the compatibility beteween points (i) and (ii) can also be considered from a geometric approach. It is known that superintegrability in compact manifolds leads to closed trajectories and implies periodicity. This is the case of the Harmonic oscillator where T^*Q is foliated by the n-dimensional tori T^n defined by $E_r = $ constant. The vector field Γ_H

is tangent to this foliation and, as the leaves are compact, the integral curves of Γ_H must be closed. This property can only be satisfied if ω_r/ω_s is rational. However, if we consider the time-dependent suspension $\widetilde{\Gamma}_H$ defined in $T^*Q \times \mathbb{R}$ then we move out of the compact tori. The time-dependent phase space $T^*Q \times \mathbb{R}$ is foliated by $(n+1)$-dimensional cylinders $T^n \times \mathbb{R}$, the vector field $\widetilde{\Gamma}_H$ is tangent to this foliation but, as in this case the leaves are non-compact, there is no additional condition and the time-dependent integrals I_r^t are always well defined. The integral curves of $\widetilde{\Gamma}_H$ are open curves that wind up the axis $\mathbb{R}$.

Acknowledgments

We thank J F Cariñena for comments on the theory of superintegrable systems and hidden symmetries. This paper is part of a project supported by grants PB-96-0717 and AEN-97-1680 (DGICYT, Madrid).

References

[1] Henon M 1983 Numerical exploration of Hamiltonian systems *Chaotic Behaviour of Deterministic Systems* ed G Iooss, R H Helleman and R Stora (Amsterdam: North-Holland)
[2] Fris T I, Mandrosov V, Smorodinsky Y A, Uhlir M and Winternitz P 1965 *Phys. Lett.* **16** 354–6
[3] Makarov A A, Smorodinsky Y A, Valiev K and Winternitz P 1967 *Nuovo Cimento* **52** 1061–84
[4] Wojciechowski S 1983 *Phys. Lett.* A **95** 279–81
[5] Chetouani L, Guechi L and Hammann T F 1987 *Phys. Lett.* A **125** 277–81
[6] Carpio-Bernido M V and Bernido C C 1987 *Phys. Lett.* A **134** 395–9
[7] Hietarinta J 1987 *Phys. Rep.* **147** 87–154
[8] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Basle: Birkhäuser)
[9] Kibler M and Winternitz P 1990 *Phys. Lett.* A **147** 338–42
[10] Evans N W 1990 *Phys. Rev.* A **41** 5666–76
[11] Evans N W 1990 *Phys. Lett.* A **147** 483–6
[12] Evans N W 1991 *J. Math. Phys.* **32** 3369–75
[13] del Olmo M A, Rodríguez M A and Winternitz P 1993 *J. Math. Phys.* **34** 5118–39
[14] del Olmo M A, Rodríguez M A and Winternitz P 1996 *Fortschr. Phys.* **44** 199–233
[15] Calzada J A, del Olmo M A and Rodríguez M A 1997 *J. Geom. Phys.* **23** 14–30
[16] Grosche C, Pogosyan G S and Sissakian A N 1995 *Fortschr. Phys.* **43** 453–521
[17] Grosche C, Pogosyan G S and Sissakian A N 1995 *Fortschr. Phys.* **43** 523–63
[18] Salas-Brito A L, Núñez-Yépez H N and Martínez-Romero R P 1997 *Int. J. Modern Phys.* **12** 271–6
[19] Rañada M F 1997 *J. Math. Phys.* **38** 4165–78
[20] Gonera C 1998 *J. Phys.* A **31** 4465–72
[21] Rañada M F 1999 Superintegrability of the Calogero–Moser system: Constants of motion, master symmetries, and time-dependent symmetries *J. Math. Phys.* to be published
[22] Tabor M 1989 *Chaos and Integrability in Nonlinear Dynamics* (New York: Wiley)
[23] Fradkin D M 1965 *Am. J. Phys.* **33** 207–11
[24] Abraham R and Marsden J 1978 *Foundations of Mechanics* (Reading, MA: Benjamin)
[25] Marmo G, Saletan E J, Simoni A and Vitale B 1985 *Dynamical Systems: A Differential Geometric Approach* (Chichester: Wiley)
[26] Libermann P and Marle C M 1987 *Symplectic Geometry and Analytical Mechanics* (Dordrecht: Reidel)
[27] de León M and Rodrigues P R 1989 *Methods of Differential Geometry in Analytical Mechanics* (Amsterdam: North-Holland)
[28] Crampin M 1983 *J. Phys. A: Math. Gen.* **16** 3755–72